

Quantum Entanglement in Many-body Systems

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(Dated: December 13, 2004)
(Updated: January 6, 2005)*

Abstract

This article is intended to be a pedagogical review of the recent studies of quantum entanglement in many-body systems. The basic concepts of entanglement for pure and mixed states are introduced in an intuitive approach. We will see that in the weakly interacting BEC and BCS superconductivity theories, the pair of particles with momentum k and $-k$ are entangled but they are not correlated with other pairs. The entanglement vanishes when the interactions go to zero. The debates on the possible relationship between quantum phase transition and quantum entanglement is mentioned. My personal perspective on this issue is discussed and illustrated with a simple example.

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Term essay for the course Physics 598: Emergent States of Matter, Fall 2004. Instructor: Nigel Goldenfeld
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1 Introduction

What is the relationship between quantum entanglement and the emergent state of matter? The main theme of the whole course is that emergent phenomena appear when individual particles are brought to *interact* together. The interaction among these particles play a major role in determining the emergent behavior of the system. Interactions, however, induce quantum entanglement. In this sense, quantum entanglement may be considered as an emergent phenomenon. Unfortunately, our understanding of quantum entanglement is still very limited. In fact, quantum entanglement has been quantified properly only for special cases, namely bipartite systems. Many-body quantum entanglement is still an open problem. This article is to review the recent studies of (bipartite) quantum entanglement in many-body systems, but it is by no means a comprehensive review. The purpose here is purely pedagogical that the concepts of entanglement and the entanglement properties of many-body systems are introduced to those curious readers having no experience in this field.

Quantum entanglement is a non-local correlation between two or more quantum systems. This type of correlation is purely quantum mechanical. This non-local property of physical systems was strongly criticized by Albert Einstein, who believed quantum theory is incomplete due to such non-locality. In 1935, Einstein, Podolsky, and Rosen [1] proposed a thought experiment to argue that there may be some hidden variables not yet discovered in the quantum theory. There was no way to exclude the possibility of hidden variables theory until 1964, John Bell proposed a set of inequalities that could test the existence of hidden variables experimentally. Experimental results, of course, “violated” the Bell’s inequalities and thus supported the non-locality of the quantum theory. For a recent review on Bell’s inequalities and hidden variable theories, see for example [3] and the references therein. Here I would like to emphasize that although quantum entanglement is a non-local property, the generation of entanglement requires interaction which must be local. Therefore, we cannot entangle anything on Mars without sending something there.

To illustrate the non-locality of entanglement, consider Alice and Bob sharing the following entangled state of two spins

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B) \quad . \quad (1)$$

If we measure one of the spins, once we obtain the result from one spin, say spin-up $|\uparrow\rangle$, then the state of another spin will be determined *immediately* (it must be spin-down $|\downarrow\rangle$). Moreover, if they choose to measure the state in another direction, say x -direction, they will still get anti-correlated results. This is true no matter how far apart these two spins are. What is perhaps more amazing is as follow: suppose Alice applies a local operation of spin-flip to her spin, usually denoted by the Pauli matrix σ_x , then the state becomes

$$|\psi'\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_A |\downarrow\rangle_B + |\uparrow\rangle_A |\uparrow\rangle_B) \quad . \quad (2)$$

Now if Alice measured a spin-up state, then Bob’s spin must be spin-up as well. The use of this non-local property of entanglement is the basic idea behind the dense coding protocol in

quantum information theory. Other applications of entanglement include quantum teleportation, quantum cryptography and quantum computation. For an elementary introduction, see for example Nielsen and Chuang [4]. Apart from the non-local property, quantum entanglement is now regarded as a resource, like energy, for doing tasks that are either more efficient or inaccessible from classical theories. For example, one can construct any state given a set of singlet states and the use of local operations and classical communications. We will talk about this more in the next section.

In this article, we will focus on the studies of entanglement in many-body systems. However before we can appreciate how to find these non-local correlations, we need to know how (bipartite) entanglement can be quantified operationally.

2 Quantifying Entanglement

Quantum entanglement, representing the non-local correlations between quantum systems, is an important part of quantum information theory. Unfortunately, entanglement can now be quantified properly only for bipartite systems. As far as I know, there is still no consensus in how multipartite entanglement should be quantified [5]. Nonetheless, bipartite entanglement are quantified differently for two types of states, pure and mixed states. A pure state of two systems is a complete description of the whole system, where there is no quantum correlation between these two systems with others. On the other hand, a mixed state appears when one of both members has correlation with other systems. Operationally, if the (reduced) density matrix of system ρ satisfy $\rho^2 = \rho$, then it is a pure state, otherwise, it is a mixed state.

2.1 Pure State Entanglement

A pure state between two systems A and B can be represented completely by the products of their bases

$$|\psi\rangle = \sum_{jk} a_{jk} |e_j\rangle_A |e_k\rangle_B \quad . \quad (3)$$

A pure state is unentangled, or called product state, if it can be decomposed into a product state

$$|\psi\rangle = |\varphi\rangle_A |\phi\rangle_B \quad . \quad (4)$$

It is known that, for pure states, all entangled states violate some Bell's type inequalities [7], but product states do not. Fortunately, there exists an operational way that can help us to determine whether a state can be decompose into a product state. This method is the (famous?) Schmidt decomposition, for a good treatment, see for example John Preskill's lecture notes [8]. For any two systems of a pure state, one can always decompose the state into a unique Schmidt form

$$|\psi\rangle = \sum_j \lambda_j |u_j\rangle_A |\tilde{u}_j\rangle_B \quad , \quad (5)$$

which is a *single* sum over modes. The main features are that (a) the bases are orthogonal and normalized, i.e. $\langle u_j | u_k \rangle = \langle \tilde{u}_j | \tilde{u}_k \rangle = \delta_{jk}$; (b) the maximum possible number of terms are constricted by the size of the smaller system. For example, if we consider a two-level atom coupled to a cavity photon mode, which is infinite dimensional, then the maximum number of terms is still two; (c) for a product state, there can be one and only one term only!

The (bipartite) entanglement for pure states are quantified by the von Neumann entropy defined by

$$S(\rho_x) = -\text{tr} \rho_x \log_2 \rho_x \quad , \quad (6)$$

where ρ_x is the reduced density matrix of system A or B . One can easily show that $S(\rho_A) = S(\rho_B)$. In fact, the square of Schmidt coefficients λ_j^2 , which can be made real and positive, are exactly the eigenvalues of the density matrix. Since the trace of the reduced density matrix must be equal to unity, we have the normalization condition

$$\sum_j \lambda_j^2 = 1 \quad , \quad (7)$$

which is consistent with the one required in (5). Consequently, the entropy can be expressed as

$$S(\rho) = - \sum_j \lambda_j^2 \log_2 \lambda_j^2 \quad , \quad (8)$$

which is minimum (zero) for product states where there is only one non-vanishing $\lambda = 1$, and maximum when all the λ 's are the same. Therefore, the state mentioned earlier in (1) is a maximally entangled state.

There are many important properties of the von Neumann entropy (see for example in [9]) to justify it to be a natural entanglement measure. The most important ones are (a) the entropy is not changed under local unitary operations $U = U_A \otimes U_B$; (b) the entropy cannot be increased by non-unitary operations such as measurement; (c) for any two distinct bipartite pure states of the same system $|x\rangle$ and $|y\rangle$, one can convert m copies of $|x\rangle$ to n copies of $|y\rangle$ using local operations and classical communications, in the limit of large m . Moreover, the ratio m/n approaches $S(y)/S(x)$ asymptotically. For example, suppose $|x\rangle$ is the maximally entangled state in (1), which means $S(x) = 1$, then the number of $|x\rangle$ needed to create n copies of $|y\rangle$ state is $m = nS(y)$. The last property is also an important concept that can be generalized to quantify entanglement for mixed states, as we shall discuss later in the next section.

With first look, readers may find it quite abstract and hard to grasp the idea of quantifying entanglement as resource. One may even question whether is it practical in doing such conversion physically (since asymptotic limit is usually needed). However, this reminds me of the definition of ‘‘potential’’ defined in the classical electromagnetic theory that the potential at a point is defined to be the mechanical work required to bring a unit positive point charge from infinity to that point. Moreover, the total energy of a charge configuration is ‘‘calculated’’ by thinking of these charges are brought from infinity little by little. Therefore, these definitions are just for conceptual purposes only. For practical operations, one seldom needs them.

2.2 Mixed State Entanglement

For mixed states, the von Neumann entropy is no longer a good measure of entanglement. The reason is clear by considering the following example: suppose A (B) and A' (B') are entangled and form a pure state. If we consider the system consisting only of A and B , which must be a mixed state and independent of each other, the entropies $S(\rho_A)$ and $S(\rho_B)$ are however non-zero and $S(\rho_A) \neq S(\rho_B)$ generally.

In fact, for pure states we mentioned that two systems are not entangled if the state can be decomposed into a product state (cf eq.(4)). For mixed states, a similar definition can be applied in the reduced density representation. For any mixed bipartite states, it is unentangled if and only if the reduced density matrix can be decomposed as the following form:

$$\rho = \sum_j p_j |\alpha_j\rangle \langle \alpha_j| \otimes |\beta_j\rangle \langle \beta_j| \quad , \quad (9)$$

which represents nothing but a set of product states. Otherwise, it is entangled. The remaining task is to properly quantify the entanglement.

In this article, we will focus on one type of entanglement measure called *entanglement formation*, which is motivated by the property (c) mentioned in the last section for the von Neumann entropy. Following the descriptions in [4, 9, 10], suppose Alice and Bob are given n copies of a particular mixed state ρ , which can be decomposed into the following form:

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j| \quad , \quad (10)$$

where $|\psi_j\rangle$ is some normalized entangled state between the two systems. It is not necessary for $|\psi_j\rangle$ to be orthogonal to each other, but the coefficients p_j have to be real and positive. With $\text{tr}\rho = 1$, one immediately see that $\sum_j p_j = 1$. To realize the mixed state in (10), one possible way is to regard the state to be a mixture consisting of pure states $|\psi_j\rangle$ weighted by p_j . To create n copies of ρ , we need asymptotically $n\sum_j p_j S(\psi_j)$ of copies of maximally entangled states. However, depending on how the mixed state is decomposed, this number would change. The entanglement formation $E_f(\rho)$ is defined by the minimum cost of creating such a mixed states

$$E_f(\rho) = \min \sum_j p_j S(\psi_j) \quad , \quad (11)$$

where the minimization is over all possible realization of the mixed state by the pure state decompositions. There are many subtle points regarding such definition of entanglement formation. For example, one can show that E_f cannot be increased by local operations and classical communications. Further details can be found for example in [9, 10].

Analytic Solution to Entanglement Formation

The definition of entanglement formation, like many other definitions that quantify entanglement, usually requires numerical minimizations and is thus not very useful in particular for many-body systems. However, for systems consisting of a pair of qubits (i.e. two level systems), Wootters (see [10] and the references therein) found an analytic solution called concurrence $C(\rho)$ that is closely related to the entanglement of formation

$$E_f(\rho) = \mathcal{E}(C(\rho)) \quad , \quad (12)$$

where

$$\mathcal{E}(C) = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \quad (13)$$

and $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. The analytic solution for concurrence is given by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \quad , \quad (14)$$

where the λ_j 's are the eigenvalues of the matrix $(\rho\tilde{\rho})^{1/2}$ and $\tilde{\rho}$ is the spin-flipped density matrix defined by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \quad , \quad (15)$$

with the complex conjugate taken in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Therefore, to calculate the entanglement between a pair of qubits, one just needs to calculate the concurrence. The form of concurrence given in (14) is perhaps much simpler than numerical minimizations. A further simplification from this can be made by exploiting symmetries and will be used in the last section.

3 Entanglement in Many-Body Systems

Here we will begin our discussion of the entanglement in Many-body systems. For pedagogical purpose, I will start with a very good example of a pair of coupled harmonic oscillators discussed by Vedral [11] to emphasize that entanglement depends strongly in the basis being studied. Moreover, the result of this example can be used directly for the system of Bose-Einstein Condensate.

3.1 Interacting Harmonic Oscillators

Suppose we consider a pair of non-interacting harmonic oscillators described by the Hamiltonian

$$H = \hbar(\omega - \lambda)b_1^\dagger b_1 + \hbar(\omega + \lambda)b_2^\dagger b_2 \quad , \quad (16)$$

where b_j satisfies the standard commutation relations for bosons. After a canonical transformation: $a_1 = (b_1 + b_2^\dagger)/\sqrt{2}$ and $a_2 = (b_1 - b_2^\dagger)/\sqrt{2}$, the Hamiltonian becomes

$$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2) + \hbar\lambda(a_1^\dagger a_2 + a_2^\dagger a_1) \quad . \quad (17)$$

The ground state of (17) can be expanded by the eigenstates $|n_b\rangle_{1,2}$ of (16) (see also [12]),

$$|0_a\rangle \propto \sum_n \left(-i \tanh \frac{4\lambda}{\omega}\right)^n |n_b\rangle_1 |n_b\rangle_2 \quad . \quad (18)$$

Note that it is already in the Schmidt form. Consequently, the entropy of entanglement can be evaluated easily as (here the “ \log_2 ” is replaced by “ \ln ”, which is just a convention)

$$S = \frac{2 \ln(\coth \eta)}{\coth^2 \eta - 1} - \ln(1 - \tanh^2 \eta) \quad , \quad (19)$$

where $\eta \equiv 4\lambda/\omega$. In the limit $\lambda \rightarrow 0$, we have $S \rightarrow 0$ as expected. It is also interesting to express the entropy in terms of “effective temperature” by a change of variables $\tanh \eta = \exp(-\hbar\omega/2k_B T)$, then the entropy becomes

$$S = \frac{\hbar\omega}{k_B T} \frac{1}{e^{\beta\hbar\omega} - 1} - \ln(1 - e^{-\beta\hbar\omega}) \quad . \quad (20)$$

3.2 Bose-Einstein Condensates

Next we generalize the above example to study the ground state entropy of weakly interacting Bose-Einstein condensates [11]. The Hamiltonian is given by the standard form

$$H = \sum_k \varepsilon_k a_k^\dagger a_k + U_0 \sum_{k_1, k_2, q} a_{k_1+q}^\dagger a_{k_2-q}^\dagger a_{k_2} a_{k_1} \quad . \quad (21)$$

The standard trick to diagonalize it is through the Bogoliubov transformation $a_k = u_k b_k + v_k b_{-k}^\dagger$ and $a_k^\dagger = u_k b_k^\dagger + v_k b_{-k}$. Under the standard approximation of macroscopic ground state occupation, the new Hamiltonian becomes

$$H = E_0 + \sum_{k \neq 0} E(k) b_k^\dagger b_k \quad . \quad (22)$$

Expanding the ground state by the new basis, it can be expressed as

$$|0\rangle = \prod_{k \neq 0} \frac{1}{u_k} \sum_{n=0}^{\infty} \left(-\frac{v_k}{u_k} \right)^n |n_k\rangle |n_{-k}\rangle \quad . \quad (23)$$

Therefore, it is apparent that only the modes k and $-k$ are entangled. It is possible to evaluate the entropy for each mode as

$$S_k = \frac{\ln(u_k/v_k)^2}{(u_k/v_k)^2 - 1} - \ln\left(1 - (v_k/u_k)^2\right) \quad . \quad (24)$$

The total entropy is just a sum over all modes, $S = \sum_k S_k$. The form of the entropy is very similar to that obtained in (19). Therefore, we should expect that in the weak interaction limit, $U_0 \rightarrow 0$, the entropy should go to zero. Moreover the condensate mode is unentangled from the rest of the system.

3.3 Bardeen-Cooper-Schrieffer Superconductivity

Here we consider the ground state entanglement of the BCS theory of superconductivity. An excellent reference for this topic may be found in [13]. The Hamiltonian we are considering is the standard one

$$H = \sum_{k,s} \varepsilon_k n_{k,s} + \sum_{k,k'} \langle k', -k' | V | k, -k \rangle a_{k'\uparrow}^\dagger a_{-k'\downarrow}^\dagger a_{-k\downarrow} a_{k\uparrow} \quad . \quad (25)$$

The BCS ground state is of the following form:

$$|0\rangle_{BCS} = \prod_k \left(u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger \right) |0\rangle \quad . \quad (26)$$

In this form, it is apparent that the electrons forming the cooper pairs are necessarily entangled, but all the cooper pairs have no quantum correlations. The entropy for each cooper mode is

$$S_k = -\lambda_k^2 \ln \lambda_k^2 - (1 - \lambda_k^2) \ln (1 - \lambda_k^2) \quad , \quad (27)$$

where $\lambda_k = |u_k|$, subject to the condition $u_k^2 + v_k^2 = 1$. The explicit solution for the ground is well-known,

$$u_k^2 = \frac{1}{2} \left(1 + \frac{\varepsilon_k}{E_k} \right) \quad , \quad v_k^2 = \frac{1}{2} \left(1 - \frac{\varepsilon_k}{E_k} \right) \quad , \quad (28)$$

where $E_k = (\varepsilon_k^2 + \Delta_k^2)^{1/2}$. The some average value of the interaction V and density of states at the Fermi level N_F , it is known that $\Delta_k \propto \exp(-1/N_F V)$ which vanishes when $V \rightarrow 0$. However, in the strong interaction limit $E_k \gg \varepsilon_k$, the cooper pairs are maximally entangled in the ground state. Furthermore, we see that the entanglement vanishes when the superconductivity vanishes, as explicitly pointed out in [13]. Lastly, it is in some sense true that the ground states in the BEC and BCS theories can be viewed in a unified point of view from the entanglement consideration that they both have entangled pairs but no entanglement between pairs.

3.4 Quantum Phase Transition and Quantum Entanglement

Quantum entanglement and quantum phase transition are both purely quantum effects. Recently, there are many attempts trying to find the relationship between quantum entanglement and quantum phase transition. As far as I know, it is still an open problem in the sense that no firm conclusion, that is universal, can be made. For a recent review, see [14, 15] and the references therein.

Here I would like to mention about the recent debates about this issue. In [14], it is claimed that under certain conditions, first-order (second-order) phase transition happens if and only if the (first order derivative of the) ground state concurrence has discontinuity or divergence. However, in [15], counter examples are found and the author argued that there is no one-to-one correspondence between the analyticity of the concurrence and the quantum phase transition. Personally, I think it is no surprising, because concurrence measures the correlation only between bipartite quantum correlations. On the other hand, it is also interesting to note that T.C. Wei and collaborators [6] studied the quantum phase transition for a quantum XY chain with the multipartite entanglement measure they invented. They found non-analyticity in the quantum phase transition regions. Perhaps, it is a good research problem to see if the multipartite entanglement measure proposed in [5] could show non-analyticity in the counter-examples proposed in [15].

My Perspectives On This Issue

Here I would like to share my personal view on the issue of (quantum) phase transition and entanglement through the discussion of the role of temperature in many-body systems on entanglement. Intuitively, in the high temperature limit, all the quantum correlations should vanish. Therefore, naively one may conclude that temperature must destroy quantum entanglement. However, I discovered that there is some situations that temperature could actually induce entanglement. Unfortunately, after some literature searching, this observation was discovered by others, see [10]. However, the explicit relation between this and phase transition has not been appreciated (as far as I know). Now I will illustrate this phenomenon with a simple example, which was first mentioned by Wang [17],

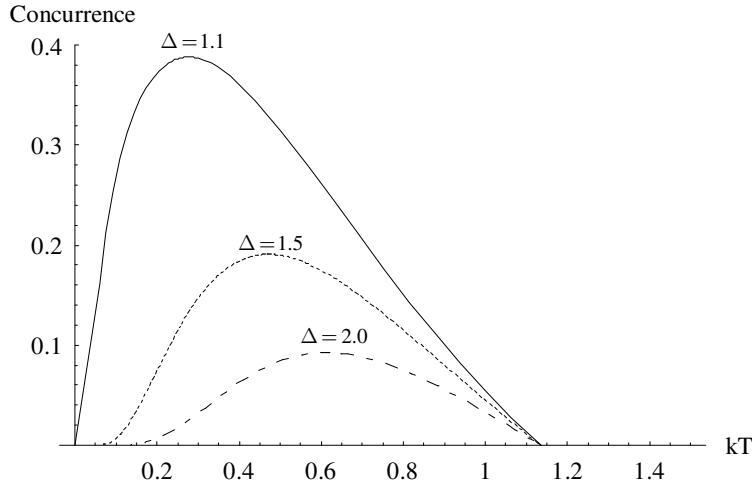


Fig. 1. Concurrence $C(\rho)$ as a function of scaled temperature kT for different values of external field Δ . For large enough Δ , the entanglement is increased by the increase of temperature.

Consider two spins (dimer) interacting under the Hamiltonian

$$H = \sigma_x^1 \sigma_x^1 + \sigma_y^2 \sigma_y^2 + \frac{\Delta}{2} (\sigma_z^1 + \sigma_z^2) \quad , \quad (29)$$

where $\Delta > 0$. Suppose the dimer is in thermal equilibrium and therefore in a mixed state, the density matrix can be written as

$$\rho = \sum_k \frac{1}{Z} e^{-\beta E_k} |e_k\rangle \langle e_k| \quad , \quad (30)$$

where $Z = \sum_k e^{-\beta E_k} = \frac{1}{2} (\cosh \beta + \cosh \Delta)^{-1}$. The eigenvalues $E_k = \{\pm 1 \pm \Delta\}$ and the eigenvectors $|e_k\rangle$ can be easily solved. In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density matrix can be written as

$$\rho = \frac{1}{Z} \begin{pmatrix} e^{-\beta E_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2} (e^{-\beta E_3} + e^{-\beta E_2}) & \frac{1}{2} (e^{-\beta E_3} - e^{-\beta E_2}) & 0 \\ 0 & \frac{1}{2} (e^{-\beta E_3} - e^{-\beta E_2}) & \frac{1}{2} (e^{-\beta E_3} + e^{-\beta E_2}) & 0 \\ 0 & 0 & 0 & e^{-\beta E_4} \end{pmatrix} \quad , \quad (31)$$

where $E_1 = -\Delta$, $E_2 = -1$, $E_3 = 1$ and $E_4 = \Delta$. The concurrence of the simple form above [16] can be evaluated easily as

$$\begin{aligned} C(\rho) &= \frac{2}{Z} \max \left\{ \frac{1}{2} |e^{-\beta E_3} - e^{-\beta E_2}| - e^{-\beta(E_1+E_4)/2}, 0 \right\} \\ &= \frac{2}{Z} \max \{ \sinh \beta - 1, 0 \} \quad . \end{aligned} \quad (32)$$

The results are shown in Figure 1. We see that for sufficiently large Δ , the concurrence and hence entanglement can be increased by the increase of temperature. The qualitative

explanations to this result is very simple but non-trivial. It is analogous to the (second order) phase transition. First of all, the eigenvectors corresponding to the above eigenvalues are

$$\begin{aligned} |e_1\rangle &= |00\rangle \quad , \quad |e_4\rangle = |11\rangle \\ |e_2\rangle &= \frac{1}{2}(|01\rangle - |10\rangle) \quad , \quad |e_3\rangle = \frac{1}{2}(|01\rangle + |10\rangle) \quad . \end{aligned} \quad (33)$$

When $\Delta < 1$, the ground state is the singlet state $|e_2\rangle$, which is maximally entangled at $T = 0$. However, when $\Delta > 1$, the ground state becomes $|e_1\rangle$, which is a product state and should have no entanglement at all at $T = 0$. Thus, the level crossing is in some sense a phase transition.

Therefore, I think that the quantum phase transition (by changing Δ), or simply phase transition (by changing T), and quantum entanglement are not necessarily related. It depends on the entanglement (either bipartite or multipartite) between the two states being crossed. Unless the two eigenstates have very different entanglement, there should be no change in entanglement.

4 Conclusions

In conclusion, the basic concepts of entanglement for pure and mixed states are introduced. Then the applications of the von Neumann entropy are discussed for many-body systems including BEC and BCS conductors. The current debates in quantum phase transition is addressed. A simple system is discussed to illustrate the possible direction to resolve the quantum phase transition debates.

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